A Law of Large Numbers for Slot Machines

In this paper we apply Bernoulli's Theorem to prove a law of large numbers for slot machines.

A slot machine returns different amounts in prizes .For example, the $1 Red White and Blue slot machine described on The Wizard of Odds website can return a prize of $1, $2, $5, $10, $20, $25, $40, $50, $80, $150, $200, $1,199, or $2,400 for a total of 13 possible returns (not counting zero).

Let K be the number of different possible returns. For the Red White and Blue slot machine, K = 13.

Let R_1 , R_2 , ... , R_K be the different possible returns

Let P_1 , P_2 , ... , P_K be their probabilities.

Let M_i be the number of times in N trials that a return of R_i occurs.

Let:

Event A_1 be $\left|\frac{M_1}{N} - P_1\right| < \frac{\varepsilon}{KR_1}$

Event A_2 be $\left|\frac{M_2}{N} - P_2\right| < \frac{\varepsilon}{KR_2}$

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Event A_K be $\left|\frac{M_K}{N} - P_K\right| < \frac{\varepsilon}{KR_K}$

If events A_1 , A_2 , ... , A_K all occur,
then:

$$\left|\frac{R_1 M_1}{N} - R_1 P_1\right| < \frac{\varepsilon}{K}$$

$$\left|\frac{R_2 M_2}{N} - R_2 P_2\right| < \frac{\varepsilon}{K}$$

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$$\left|\frac{R_K M_K}{N} - R_K P_K\right| < \frac{\varepsilon}{K}$$

all occur and $\left|\frac{R_1 M_1 + R_2 M_2 + \ldots + R_K M_K}{N} - (R_1 P_1 + R_2 P_2 + \ldots + R_K P_K)\right| < \varepsilon$
will also occur, where $R_1M_1+R_2M_2+ \ldots +R_KM_K$ is the total return in N trials and $R_1P_1+R_2P_2+ \ldots +R_KP_K$ is the expected return per trial.

So | Actual Return per trial - Expected Return per trial | < ε

occurs if the events A_1 , A_2 , ... , A_K all occur .

Lemma
Let $P(A_1)$, $P(A_2)$, ... , $P(A_K)$ be the probabilities that A_1 , A_2 , ... , A_K occur, then the probability $P(A_1A_2 \ldots A_K)$ that all the events occur is $\geq P(A_1)+ P(A_2)+ \ldots +P(A_K) - (K-1)$.

(See proof below)

By Bernoulli's Theorem, the probability that a given one of the events occurs will be greater than $1 - \frac{\eta}{K}$ if N is sufficiently large. So if N is sufficiently large, $P(A_1)$, $P(A_2)$, ... , $P(A_K)$ will each be greater than $1 - \frac{\eta}{K}$ So by the lemma, if N is sufficiently large, the probability that all the events occur will be $\geq K(1- \frac{\eta}{K})-(K-1) =$

$1 - \eta$

.

So if N is sufficiently large, the probability that :

| Actual Return per trial - Expected Return per trial | $< \varepsilon$ will be

be greater than $1 - \eta$. The theorem is proved.

Proof of the lemma

Consider the case $k = 2$.

Let $\overline{A_1}$ be the event that A_1 does not occur and let $\overline{A_2}$ be the event that A_2 does not occur.

$$P(A_1) = P(A_1A_2) + P(A_1 \overline{A_2})$$

$$P(A_2) = P(A_1A_2) + P(\overline{A_1} A_2)$$

$$P(A_1A_2) + P(A_1 \overline{A_2}) + P(\overline{A_1} A_2) + P(\overline{A_1}\, \overline{A_2}) = 1$$

So $P(A_1) + P(A_2) - 1 = P(A_1A_2) - P(\overline{A_1}\, \overline{A_2})$.

So = $P(A_1A_2) \geq P(A_1) + P(A_2) - 1$

with equality only if $P(\overline{A}_1\ \overline{A}_2) = 0$.

If $K > 2$, then just like before there will be probability K-tuples. For example, if $K = 4$, one of the possible probability K-tuples would be $P(A_1 \overline{A}_2 \overline{A}_3 A_4)$. This is the probability that events A_1 and A_4 both occur and that events A_2 and A_3 don't occur. This probability k-tuple is said to have two bars.
The sum of all the possible probability K-tuples is one. So when K - 1 is subtracted, each probability K-tuple is subtracted K-1 times.
$P(A_i)$ is the sum of all the probability K-tuples in which A_i does not have a bar over it. Any probability K-tuple which has a bar over $A_i$is not included.
So every probability K-tuple with one bar will not be included in one of the $P(A_i)$'s, so it will only be counted K - 1 times in the sum $P(A_1)+P(A_2)+ ....+P(A_K)$. So since it is being subtracted K - 1 times, no probability K-tuple with one bar will be in the answer.
If a probability K-tuple has two bars, it won't be included in two of the $P(A_i)$'s, so it will only be counted K - 2 times in the sum $P(A_1)+P(A_2)+ ....+P(A_K)$. So since it is being subtracted K - 1 times, it will occur once in the answer as a negative amount.
If a probability K-tuple has three bars, it won't be included in three of the $P(A_i)$'s, so it will only be counted K - 3 times in the sum $P(A_1)+P(A_2)+ ....+P(A_K)$. So since it is being subtracted K - 1 times, it will occur in the answer twice, each time as a negative amount.
The only probability K-tuple to occur K times is the one with no bars, $P(A_1A_2 ...A_K)$. So since it is being subtracted K - 1 times, it will occur in the answer once as a positive amount.
So $P(A_1)+P(A_2)+ ....+P(A_K) - (K - 1) =$
$P(A_1A_2 ...A_K)$ - the sum of all probability K-tuples with two bars - twice the sum of all probability K-tuples with three bars and so on.
So $P(A_1A_2 ...A_K) \geq P(A_1)+P(A_2)+ ....+P(A_K) - (K - 1)$
with equality only if all the probability K-tuples with two or more

bars have a probability of zero.

The lemma is proved.

Comment

The Red White and Blue slot machine described on the Wizard of Odds website has an expected return of .8658 and a variance of 82.12.
Using Chebyshev's Inequality, the N needed to guarantee that the probability that
the | Actual Return per trial - Expected Return per trial | < .05 will be greater than 1 - .05 is 656,960.

Combining the Bernoulli theorem formulas with the proof given above one gets an N of over two trillion.

So for the Slot Machine problem Chebyshev's proof of the Law of Large Numbers is much better.

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